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A temperature adjustment process in a Boussinesq fluid via a buoyancy-induced meridional circulation

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The existence of a new time scale $t = \sigma^{\frac{1}{4}}(L^2/kN)^{\frac{1}{2}}$, where σ, k, L and N are the Prandtl number, the thermometric conductivity, a typical length and the Brunt–Väisälä frequency, respectively, is clarified for a temperature adjustment process of a Boussinesq fluid in a circular cylinder.

1. Introduction and summary

Let us consider a Boussinesq fluid at rest in a circular cylinder whose axis of symmetry is parallel to the gravitational force. The temperature at the top of the cylinder is higher than that at the bottom to make the temperature distribution thermally stable. The material of the cylinder has a large thermometric conductivity so that any change of the top or the bottom temperature is relaxed (within the material) in a time small compared with the time scale of hydrodynamical processes. At a certain instant, the top and the bottom temperatures are changed abruptly by small amounts of equal magnitude but opposite sign to change the temperature distribution of the side wall. Our problem is to study the response of the fluid to this abrupt change of the wall temperature. The linearized theory is applied for the case where the Prandtl number is of order unity.

There exists an analogy between dynamical processes in a rotating fluid and those in a stratified fluid (Veronis 1970). We are already familiar with the spin-down process in a rotating fluid (Greenspan & Howard 1963), in which a meridional circulation pumped by the Ekman boundary layer redistributes the angular momentum. In this paper, we want to demonstrate the existence of a meridional circulation pumped by a side-wall boundary layer; this meridional circulation redistributes the fluid temperature to bring about a new state of stratification. This gives us a process in a stratified fluid which is analogous to the spin-down process in a rotating fluid.

Before going directly to the discussion of the mathematical treatment, let us consider relevant physical processes. A quasi-steady side-wall boundary layer is formed within a few periods of the Brunt–Väisälä oscillation after the onset of the temperature change of the wall. The thickness ϵ of the boundary layer is the hybrid thermal and viscous diffusion length for the time scale of the Brunt–Väisälä oscillation:

$$\epsilon = \sigma^{\frac{1}{4}}(k/N)^{\frac{1}{2}}, \tag{1}$$

where k is the thermometric conductivity, σ the Prandtl number and N the Brunt–Väisälä frequency. There is a vertical flow in this boundary layer because of the excess buoyancy force due to the change in the wall temperature. This buoyancy force is balanced by the viscous force:

$$\hat{\rho}_1 g \sim \rho_0 \sigma k \hat{q}/\epsilon^2, \tag{2}$$

where $\hat{\rho}_1$ is the excess density corresponding to the change of the wall temperature, g the gravitational acceleration, $\hat{q}$ the vertical velocity, ρ_0 the standard density, and carets refer to the boundary layer. It is interesting that the structure of our boundary layer is the same as that of the buoyancy layer in the rotating stratified fluid (Barcilon & Pedlosky 1967). Because of the continuity of the flow, this vertical flow in the boundary layer pumps a meridional circulation in the non-conducting inviscid region outside the boundary layer. The meridional velocity is of the order of the velocity of the boundary-layer flow scaled down by the ratio of the boundary-layer thickness to the typical length L of the cylinder:

$$q \sim \hat{q}\epsilon/L. \tag{3}$$

This meridional circulation transports the thermal energy of each of the fluid particles. Because of the basic thermal stratification, this convection brings about a new state of stratification. Let us consider, for example, the case in which the wall temperature is changed so that the stratification is increased, that is, the case in which the top temperature is raised and the bottom temperature is lowered. An upward flow occurs in the upper half of the side-wall boundary layer and a downward flow in the lower half. This boundary-layer flow sucks up the outer fluid and induces a meridional circulation flowing downward in the upper half and upward in the lower half of the main body of the non-conducting inviscid region. The coupling of this meridional circulation with the basic stratification increases the overall stratification. This process is consistent with the change in the side-wall temperature. The time rate of change of the fluid temperature T_1 by this thermal convection is

$$T_1/t \sim q\beta, \tag{4}$$

where t is the time scale of this process and β is the basic temperature gradient. The expression for t given in the abstract is readily obtained from these relations by taking into account the usual density–temperature relation of a Boussinesq fluid,

$$\hat{\rho}_1 \sim \rho_0 \alpha \hat{T}_1, \tag{5}$$

where α is the coefficient of thermal expansion, the relation

$$\hat{T}_1 \sim T_1, \tag{6}$$

between the temperature perturbations inside and outside the boundary layer, and the expression for the Brunt–Väisälä frequency,

$$N = (\alpha g \beta)^{\frac{1}{2}}. \tag{7}$$

The validity of our description of the physical processes depends on the existence of the side-wall boundary layer being established within the time scale t.

This requires the boundary layer to be thin compared with the typical length of the cylinder, i.e.

$$E = \epsilon/L = \sigma^{\frac{1}{4}}(k/L^2N)^{\frac{1}{2}} \ll 1. \tag{8}$$

It is interesting that E is also the ratio of our time scale to the thermal diffusion time L^2/k. Thus, our time scale is small compared with the thermal diffusion time.

2. Basic equations

In cylindrical co-ordinates the basic equations of a meridional flow in a Boussinesq fluid are

$$\left(\frac{\partial}{\partial t}-\sigma kL\right)L\psi = -\alpha g\frac{\partial T_1}{\partial r}, \tag{9}$$

$$\left(\frac{\partial}{\partial t}-k\Delta\right)T_1 = \frac{\beta}{r}\frac{\partial(r\psi)}{\partial r}, \tag{10}$$

where

$$q_r = \frac{\partial\psi}{\partial z}, \quad q_z = -\frac{1}{r}\frac{\partial(r\psi)}{\partial r}, \tag{11}$$

$$\Delta = \frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{\partial^2}{\partial z^2}, \quad L = \Delta-\frac{1}{r^2}, \tag{12}$$

r and z being the radial and vertical co-ordinates, respectively, and q_r and q_z the corresponding velocity components. The basic stationary state is

$$T_B = T_0(1+\beta z/T_0). \tag{13}$$

The perturbation on the stationary state is expressed as

$$T = T_B+T_1, \tag{14}$$

with the meridional velocity itself taken as a perturbation. Equations (9) and (10) are derived by substituting these expressions into the original basic equations, neglecting the second-order terms with respect to the perturbation, and eliminating the pressure from the resulting equations.

The initial and boundary conditions are as follows:

$$T_1 = 0, \quad \psi = 0 \quad \text{for} \quad 0 \leqslant z \leqslant L, \quad r \leqslant \tilde{r}_0 L, \quad t \leqslant 0, \tag{15a, b}$$

$$T_1 = 2KT_0(z/L-\tfrac{1}{2}), \quad \psi = \partial\psi/\partial r = 0 \quad \text{for} \quad 0 \leqslant z \leqslant L, \quad r = \tilde{r}_0 L, \quad t > 0, \tag{16a, b}$$

$$T_1 = \pm T_0 K, \quad \psi = \partial\psi/\partial z = 0 \quad \text{for} \quad z = (\tfrac{1}{2}\pm\tfrac{1}{2})L, \quad r \leqslant \tilde{r}_0 L, \quad t > 0, \tag{17a, b}$$

where $\tilde{r}_0$ is the ratio of the radius to the height of the cylinder, and K designates a perturbation on the temperature gradient of the wall. Let us introduce the following rescaling, in accordance with the physical considerations above:

$$\left.\begin{aligned} \psi &= \frac{kT_0}{\beta}\left(\frac{\alpha g\beta}{\sigma k^2}\right)^{\frac{1}{4}}\tilde{\psi}, \quad t = \frac{L}{k}\left(\frac{\sigma k^2}{\alpha g\beta}\right)^{\frac{1}{4}}\tilde{t}, \\ T_1 &= T_0\tilde{T}_1, \quad (r, z) = L(\tilde{r}, \tilde{z}). \end{aligned}\right\} \tag{18}$$

The substitution of these transformations into (9) and (10) leads to the following:

$$E^2\left(\frac{\partial}{\partial \tilde{t}}-\sigma E\tilde{L}\right)\tilde{L}\tilde{\psi} = -\sigma\frac{\partial \tilde{T}_1}{\partial \tilde{r}}, \tag{19}$$

$$\left(\frac{\partial}{\partial \tilde{t}}-E\tilde{\Delta}\right)\tilde{T}_1 = \frac{1}{\tilde{r}}\frac{\partial(\tilde{r}\tilde{\psi})}{\partial \tilde{r}}. \tag{20}$$

3. Method of solution

Let us introduce the boundary-layer decomposition of the physical quantities:

$$\tilde{T}_1 = \tilde{T}^I+\hat{T}, \quad \tilde{\psi} = \tilde{\psi}^I+\hat{\psi}, \tag{21}$$

where the suffix I refers to the inner non-conducting inviscid region and the caret to the side-wall boundary layer. The boundary-layer quantities depend on the boundary-layer variable δ instead of on $\tilde{r}$, where

$$\tilde{r} = \tilde{r}_0 - E\delta. \tag{22}$$

Substitution of this decomposition into (19) and (20) and retention of the lowest order terms with respect to E leads to

$$\frac{\partial \tilde{T}^I}{\partial \tilde{r}} = 0, \quad \frac{\partial \tilde{T}^I}{\partial \tilde{t}} = \frac{1}{\tilde{r}}\frac{\partial(\tilde{r}\tilde{\psi}^I)}{\partial \tilde{r}}, \tag{23a,b}$$

$$\partial^3\hat{\psi}/\partial\delta^3 = -\hat{T}, \quad \partial\hat{T}/\partial\delta = \hat{\psi}. \tag{24a,b}$$

The substitution of (21) and of the solution of (24) into (16) gives us the side-wall boundary condition for the inner non-conducting inviscid flow;

$$2^{\frac{1}{2}}\tilde{\psi}^I_w+\tilde{T}^I_w = 2K(\tilde{z}-\tfrac{1}{2}), \tag{25}$$

where the suffix w refers to the side wall.

The solution of (23) gives us

$$\tilde{T}^I = \tilde{T}^I_w(\tilde{z},\tilde{t}), \quad \tilde{\psi}^I = \tilde{\psi}^I_w(\tilde{z},\tilde{t})\,\tilde{r}/\tilde{r}_0. \tag{26}$$

The substitution of (26) into (23b) gives us

$$\partial\tilde{\psi}^I_w/\partial\tilde{t}+(2^{\frac{1}{2}}/\tilde{r}_0)\,\tilde{\psi}^I_w = 0. \tag{27}$$

The initial value of $\tilde{\psi}^I_w$ is obtained from the fact that $\tilde{T}^I$ vanishes at the initial instant, see (15a). Hence

$$\tilde{\psi}^I_w|_{\tilde{t}=0} = 2^{\frac{1}{2}}K(\tilde{z}-\tfrac{1}{2}). \tag{28}$$

It is interesting that the initial value of $\tilde{\psi}^I$ is not zero, which contradicts (15b). This is due to the fact that our boundary-layer decomposition is meaningful only after the establishment of the quasi-steady side-wall boundary layer. Thus, the initial instant for the inner flow is a few periods of the Brunt–Väisälä oscillation after the real initial instant. During this time the meridional circulation has already been established by the pumping mechanism of the side-wall boundary layer. This situation is also analogous to that in the spin-down process (Greenspan & Howard 1963).

The solution of (27) subject to (28) gives us

$$\tilde{\psi}^I = 2^{\frac{1}{2}}K(\tilde{z}-\tfrac{1}{2})\frac{\tilde{r}}{\tilde{r}_0}\exp\left(-\frac{2^{\frac{1}{2}}}{\tilde{r}_0}\tilde{t}\right), \tag{29}$$

$$\tilde{T}^I = 2K(\tilde{z}-\tfrac{1}{2})\left\{1-\exp\left(-\frac{2^{\frac{1}{2}}}{\tilde{r}_0}\tilde{t}\right)\right\}. \tag{30}$$

The above solution justifies the physical assumptions of § 1.

4. Concluding remarks

As is shown in (23), and in analogy with the Taylor–Proudman column in the rotating homogeneous fluid, our perturbation is independent of the horizontal co-ordinates. This situation does not change as long as the density is a function of temperature only. Because our boundary-layer variables are local with respect to the vertical co-ordinates, we believe that our conclusion has a wide range of applicability for the temperature adjustment process in the generally stratified fluid. Finally, the matching of the inner meridional circulation and of the inner temperature field to the boundary conditions on the horizontal walls is achieved via the horizontal boundary layers. However, because the existence of these horizontal layers does not affect our conclusions, we omit the discussion of these layers.

Note added in proof. After the submission of our paper to the *Journal of Fluid Mechanics,* we had a chance to read a paper by Walin (1971). He gave a general treatment of the temperature adjustment process of a Boussinesq fluid with boundaries of finite conductance. Because our side wall is a perfect conductor, the effect of the present estimate of the time scale is to complement Walin's estimates.

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